Applications of special functions

Ali Omar

1. Introduction

Let A denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in $U = \{z: |z| < 1\}$. If a function f is given by (1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,\ z \in U.$$

Let W denote the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n\ (a_n \geq 0)$$

Definition 1. ([3]) The Gaussian hypergeometric function denoted by ${}_2F_1(a,b;c;z)$ and is defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,\ |z| < 1,$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, c > b > 0$ and $c > a + b$.

It is well known (see [1]) that under the conditions $c > b > 0$ and $c > a + b$ we have

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

DEfinition 2. Let $f(z) \in W$ be of the form (4), then the Hohlov operator $F(a,b,c), (F(a,b,c): W \to W)$([2]) is defined by means of a Hadamard product below:

$$F(a,b,c)f(z) = (z_2F_1(a,b;c;z)) * f(z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n,$$

$(a,b,c \in \mathbb{N}, c \neq Z_0^-; z \in U)$

The integral representation of Hohlov operator is given by

$$
\begin{aligned}
& F(a,b,c)f(z) = \\
= \;& \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{(1-\sigma)^{c-a-b}\sigma^{b-2}}{\Gamma(c-a-b+1)} \, {}_2F_1(c-a, 1-a; c-a-b+1; 1-\sigma) f(z) \mathrm{d}\sigma, \\
& (a>0,\ b>0, c-a-b+1>0,\ f \in W, z \in U) \\
= \;& \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} I_{1,2}^{(a-2,b-2),(1-a,c-b)} f(z).
\end{aligned}
$$

FRACTIONAL CALCULUS OPERATORS FOR A CLASS OF UNIVALENT FUNCTIONS

Definition 3. A function $f(z)$ in W is in the class $W(a,b,c,\gamma,\beta)$ if it satisfies the condition

$$
\left| \frac{z(F(a,b,c)f(z))''}{z(F(a,b,c)f(z))'' - 2(1-\gamma)(F(a,b,c)f(z))'} \right| < \beta
$$

where $0 \leq \gamma < 1, 0 < \beta \leq 1, z \in U$

2. The class $W(a,b,c,\gamma,\beta)$

THEOREM 1. Let the function f be defined by (4). Then $f \in W(a,b,c,\gamma,\beta)$ if and only if

$$
\sum_{n=2}^{\infty} n[(n-1)(1-\beta) + 2\beta(1-\gamma)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n \leq 2\beta(1-\gamma),
$$

where $0 \leq \gamma < 1, 0 < \beta \leq 1$.

The result (9) is sharp for the function

$$
f(z) = z - \frac{2\beta(1-\gamma)z^n}{n[(n-1)(1-\beta) + 2\beta(1-\gamma)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}},\ n \geq 2.
$$

P r o of. Suppose that the inequality (9) holds true and $|z| = 1$. Then we obtain

$$
\begin{aligned}
& |z(F(a,b,c)f(z))''| - \beta|z(F(a,b,c)f(z))'' - 2(1-\gamma)(F(a,b,c)f(z))'| \\
= \;& \left| -\sum_{n=2}^{\infty} n(n-1) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^{n-1} \right| \\
& -\beta \left| \sum_{n=2}^{\infty} [n(n-1) - 2n(1-\gamma)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^{n-1} + 2(1-\gamma) \right| \\
\leq \;& \sum_{n=2}^{\infty} n[(n-1)(1-\beta) + 2\beta(1-\gamma)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n - 2\beta(1-\gamma) \\
\leq \;& 0
\end{aligned}
$$

Hence, by maximum modulus principle, $f \in W(a,b,c,\gamma,\beta)$.

Now suppose that $f \in W(a,b,c,\gamma,\beta)$ so that

$$\left|\frac{z(F(a,b,c)f(z))''}{z(F(a,b,c)f(z))'' - 2(1-\gamma)(F(a,b,c)f(z))'}\right| < \beta,\ z \in U,$$

then

$$|z(F(a,b,c)f(z))''| < \beta|z(F(a,b,c)f(z))'' - 2(1-\gamma)(F(a,b,c)f(z))'|$$

we get

$$\left|-\sum_{n=2}^{\infty} n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-1}\right| < \beta\left|\sum_{n=2}^{\infty} [n(n-1) - 2n(1-\gamma)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-1} + 2(1-\gamma)\right|,$$

thus

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta) + 2\beta(1-\gamma)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n \le 2\beta(1-\gamma),$$

and the proof is complete.

3. Application of the fractional calculus

Various operators of fractional calculus (that is, fractional derivative and fractional integral) have been rather extensively studied by many researchers (c.f. [5], [6], [7]). However, we try to restrict ourselves to the following definition given by Owa [4] for convenience.

DEfinition 4 (Fractional integral operator). The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}\mathrm{d}t\ (\lambda > 0)$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

DEFinition 5 (Fractional derivative operator). The fractional derivative of order λ is defined, for a function $f(z)$ by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{\mathrm{d}}{\mathrm{d}z}\int_0^z \frac{f(t)}{(z-t)^{\lambda}}\mathrm{d}t\ (0 \le \lambda < 1)$$

where $f(z)$ is constrained, and the multiplicity of $(z-t)^{-\lambda}$ is removed, as in Definition 4.
Definition 6. Under the hypothesis of Definition 5 , the fractional derivative of order $k+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{k+\lambda} f(z)=\frac{\mathrm{d}^k}{\mathrm{d} z^k} D_z^\lambda f(z)\ (0 \leq \lambda<1,\ k \in \mathbb{N}_0)$$

Next, we state the following definition of fractional integral operator given by Srivastava et. al. [8].

Definition 7. For real numbers $\alpha>0, \eta$ and δ, the fractional operator, $I_{0, z}^{\alpha, \eta, \delta}$ is defined by

$$I_{0, z}^{\alpha, \eta, \delta} f(z)=\frac{z^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^z(z-t)^{\alpha-1} F\left(\alpha+\eta,-\delta ; \alpha ; 1-\frac{t}{2}\right) f(t) \mathrm{d} t$$

where $f(z)$ is analytic function in a simply connected region of the z-plane containing the origin with order

$$f(z)=O\left(|z|^{\varepsilon}\right),\ z \rightarrow 0,$$

where $\varepsilon>\max (0, \eta-\delta)-1$,

$$F(a, b ; c ; z)=\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z$$

and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & (n=0) \\ \lambda(\lambda+1) \cdots(\lambda+n-1) & (n \in \mathbb{N})\end{cases}$$

and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log (z-t)$ to be real, when $z-t>0$

In order to prove our result concerning the fractional integral operator, we recall here the following lemma due to Srivastava et. al. [8].

LEMMA 1. Let $\alpha>0$ and $n>\eta-\delta-1$. Then

$$I_{0, z}^{\alpha, \eta, \delta} z^n=\frac{\Gamma(n+1) \Gamma(n-\eta+\delta+1)}{\Gamma(n-\eta+1) \Gamma(n+\alpha+\delta+1)} z^{n-\eta} .$$

Now making use of above Lemma 1, we state and prove the following theorem:

THEOREM 2. Let $\alpha>0, \eta<2, \alpha+\delta>-2, \eta(\alpha+\delta) \leq 3 \alpha$. If $f(z)$ defined by (4) is in the class $W(a, b, c, \gamma, \beta)$, then

$$\left|I_{0, z}^{\alpha, \eta, \delta} f(z)\right| \geq \frac{\Gamma(2-\eta+\delta)|z|^{1-\eta}}{\Gamma(2-\eta) \Gamma(2+\alpha+\delta)}\left(1-\frac{2 c \beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(1+\beta(1-2 \gamma)) a b}|z|\right)$$

and

$$\left|I_{0,z}^{\alpha,\eta,\delta}f(z)\right| \leq \frac{\Gamma(2-\eta+\delta)|z|^{1-\eta}}{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)}\left(1+\frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(1+\beta(1-2\gamma))ab}|z|\right),$$

for $z \in U_0$, where

$$U_0 = \begin{cases} U & n \leq 1 \\ U - \{0\} & n > 1 \end{cases}$$

The result is sharp and is given by

$$f(z) = z - \frac{\beta(1-\gamma)}{ab(1+\beta(1-2\gamma))}z^2.$$

P r o of. By using Lemma 1, we have

$$\begin{aligned} I_{0,z}^{\alpha,\eta,\delta}f(z) = \quad & \frac{\Gamma(2-\eta+\delta)}{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)}z^{1-\eta} \\ & -\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(n-\eta+\delta+1)}{\Gamma(n-\eta+1)\Gamma(n+\alpha+\delta+1)}a_n z^{n-\eta} \end{aligned}$$

Setting

$$Q(z) = \frac{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)}{\Gamma(2-\eta+\delta)}z^{\eta}I_{0,z}^{\alpha,\eta,\delta}f(z) = z - \sum_{n=2}^{\infty} q(n)a_n z^n$$

where

$$q(n) = \frac{(2-\eta+\delta)_{n-1}(1)_n}{(2-\eta)_{n-1}(2+\alpha+\delta)_{n-1}} \ (n \geq 2)$$

It is easily verified that $q(n)$ is non-increasing for $n \geq 2$, and thus we have

$$0 < q(n) \leq q(2) = \frac{(2-\eta+\delta)2}{(2-\eta)(2+\alpha+\delta)}.$$

Now, by the application of Theorem 1 and (19), we obtain

$$\begin{aligned} |Q(z)| \quad & \geq |z| - q(2)|z|^2\sum_{n=2}^{\infty} a_n \\ & \geq |z| - \frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(1+\beta(1-2\gamma))ab}|z|^2, \end{aligned}$$

which proves (14), and for (15) we can find that

$$
\begin{aligned}
|Q(z)| \quad &\leq |z| + q(2)|z|^2 \sum_{n=2}^{\infty} a_n \\
&\leq |z| + \frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(1+\beta(1-2\gamma))ab}|z|^2
\end{aligned}
$$

and the proof is complete.

Taking $\eta = -\alpha = -\lambda$ and $\eta = -\alpha = \lambda$ in Theorem 2 , we get two separate corollaries, which are contained in:

CoROLLARY 1. Let the function f defined by (4) be in the class $W(a,b,c,\gamma,\beta)$. Then we have

$$
\left|D_z^{-\lambda} f(z)\right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left(1 - \frac{2c\beta(1-\gamma)}{(2+\lambda)(1+\beta(1-2\gamma))ab}|z|\right)
$$

and

$$
\left|D_z^{-\lambda} f(z)\right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left(1 + \frac{2c\beta(1-\gamma)}{(2+\lambda)(1+\beta(1-2\gamma))ab}|z|\right).
$$

CoRoLLARY 2. Let the function f defined by (4) be in the class $W(a,b,c,\gamma,\beta)$. Then we have

$$
\left|D_z^{\lambda} f(z)\right| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left(1 - \frac{2c\beta(1-\gamma)}{(2-\lambda)(1+\beta(1-2\gamma))ab}|z|\right)
$$

and

$$
\left|D_z^{\lambda} f(z)\right| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left(1 + \frac{2c\beta(1-\gamma)}{(2-\lambda)(1+\beta(1-2\gamma))ab}|z|\right)
$$

4. REFERENCES

1. M. Abramowitz and I. A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications Inc., New York, 1965.
2. A. A. Attiya, Some applications of Mittag-Leffler function in the unit disk, Filomat 30(7) (2016), 2075-2081. https://doi .org/10 . 2298/FIL1607075A
3. D. Bansal and J. K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, Complex Var. Elliptic Equ. 61(3) (2016), 338-350. https://doi.org/10.1080/17476933.2015. 1079628

4. T. Bulboacă and G. Murugusundaramoorthy, Univalent functions with positive coefficients involving Pascal distribution series, Commun. Korean Math. Soc. 35(3) (2020),867 − 877. https: //doi . org/10.4134/CKMS . c190413
5. N. E. Cho, S. Y. Woo and S. Owa, Uniform convexity properties for hypergeometric functions, Fract. Calc. Appl. Anal. 5(3) (2002), 303-313.
6. P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften Series 259 , Springer Verlag, New York, 1983. [7] M. El-Deeb, T. Bulboacă and J. Dziok, Pascal distribution series connected with certain subclasses of univalent functions, Kyungpook Math. J. **59** (2019), 301-314. https://doi.org/10.5666/ KMJ . 2019.59.2.301
7. B. A. Frasin, An application of an operator associated with generalized Mittag-Leffler function, Konuralp J. Math. 7(1) (2019), 199-202.
8. B. A. Frasin, T. Al-Hawary and F. Yousef, Some properties of a linear operator involving generalized Mittag-Leffler function, Stud. Univ. Babeş-Bolyai Math. 65(1) (2020),67 − 75. https: //doi . org/10.24193/subbmath.2020.1.06
9. B. A. Frasin, T. Al-Hawary and F. Yousef, Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions, Afr. Mat. 30(1-2) (2019), 223-230. https: //doi.org/10.1007/s13370-018-0638-5
10. H. J. Haubold, A. M. Mathai and R. K. Saxena, Mittag-Leffler functions and their applications, J. Appl. Math. 2011(2011), Article ID 298628. https://doi.org/10.1155/2011/298628
11. V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series 301, Longman Scientific & Technical, Harlow, John Wiley & Sons, Inc., New York, 1994.
12. E. Merkes and B. T. Scott, Starlike hypergeometric functions, Proc. Amer. Math. Soc. 12 (1961), 885-888.
13. G. M. Mittag-Leffler, Sur la nouvelle fonction $\mathbf{E}(x)$, C. R. Acad. Sci. Paris 137(1903), 554 − 558.
14. G. Murugusundaramoorthy, Subordination results for spiral-like functions associated with the Srivastava-Attiya operator, Integral Transforms Spec. Funct. 23(2) (2012), 97-103. https: //doi. org/10.1080/10652469.2011.562501

15. G. Murugusundaramoorthy, D. Răducanu and K. Vijaya, A class of spirallike functions defined by Ruscheweyh-type q-difference operator, Novi Sad J. Math. $\mathbf{49}(2)(2019), 59-71$. https://doi. org/10.30755/NSJOM.08284
16. G. Murugusundaramoorthy, K. Vijaya and S. Porwal, Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series, Hacet. J. Math. Stat. 45(4) (2016), 1101-1107. https: //doi.org/10.15672/HJMS . 20164513110
17. G. Murugusundaramoorthy, Subclasses of starlike and convex functions involving Poisson distribution series, Afr. Mat. 28(2017), 1357-1366. https://doi.org/10.1007/s13370-017-0520-x
18. S. Porwal and M. Kumar, A unified study on starlike and convex functions associated with Poisson distribution series, Afr. Mat. $27(5)$ (2016), 1021-1027. https://doi.org/10.1007/ s13370-016-0398-z
19. M. S. Robertson, On the theory of univalent functions, Ann. of Math. (2) 37(2) (1936), 374-408.
20. H. Silverman, Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl. 172 (1993), 574-581. https://doi . org/10.1006/jmaa.1993. 1044
21. H. M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, Hypergeometric functions in the parabolic starlike and uniformly convex domains, Integral Transforms Spec. Funct. 18 $(2007), 511-520$. https://doi.org/10.1080/10652460701391324
22. L. Spaček, Contribution à la théorie des fonctions univalentes, Časopis Pro Pěstování Matematiky $\mathbf{62}(1932), 12-19$.
23. A. Swaminathan, Certain sufficient conditions on Gaussian hypergeometric functions, Journal of Inequalities in Pure and Applied Mathematics $5(4)$ (2004), Article ID 83, 10 pages.
24. A. Wiman, Über die nullstellen der funktionen $\mathbf{E}_{\alpha}(x)$, Acta Math. 29 (1905), 217-134.
25. T. R. Prabhakar, A single integral equation with a generalized Mittag – Leffler function in the kernel, Yokohama Math. J. 19 (1971), 7 – 15.
26. Salah, J. and Darus, M., A note on Generalized Mittag-Leffler function and Application, Far East Journal of Mathematical Sciences (FJMS) 48 (1), 33–46, 2011.

27. Omar Jawabreh, Ahmad Abdel Qader, Jamal Salah, Khaled Al Mashrafi, Emad Al Dein AL Fahmawee, Basel J. A. Ali, Fractional Calculus Analysis of Tourism Mathematical Model, Progr. Fract. Differ. Appl. Vol. 9, No. S1, pp. 1 -11 202) DOI: 10.18576/pfda/09S101 URL: https://www.naturalspublishing.com/Article.asp?ArtcID=26373
28. Jamal Salah, Maryam Al Hashmi and Khaled Al Mashrafi, Some propositions on mathematical models of population growth, International Journal of Latest Trends in Engineering and Technology (IJLTET)Special Issue - ICTIMESH - 2022, 64-69. URL: https://www.ijltet.org/journal_details.php?id=978&j_id=5050
29. Jamal Salah, Hameed Ur Rehman, and, Iman Al- Buwaiqi, Inclusion Results of a Generalized Mittag-Leffler-Type Poisson Distribution in the k-Uniformly Janowski Starlike and the k-Janowski Convex Functions, Mathematics and Statistics, Vol. 11, No. 1, pp. 22-27, 2023. DOI: 10.13189/ms.2023.110103
30. Jamal Salah, Riemann and Big Bang Hypotheses, NEUROQUANTOLOGY, Vol. 20, No. 16, pp. 1592-1597, 2022. DOI: 10.48047/NQ.2022.20.16. NQ880156
31. Jamal Salah, Hameed Ur Rehman, and, Iman Al- Buwaiqi, Subclasses of Spiral-Like Functions Associated with the Generalized Mittag-Leffler Function, NEUROQUANTOLOGY, Vol. 20, No. 15, pp. 4784-4799, 2022. DOI: 10.14704/NQ.2022.20.15. NQ88484
32. Jamal Salah, Maryam Al Hashmi, Hameed Ur Rehman, Khaled Al Mashrafi, Modified Mathematical Models in Biology by the Means of Caputo Derivative of A Function with Respect to Another Exponential Function, Mathematics and Statistics, Vol. 10, No. 6, pp. 1194 - 1205, 2022. DOI: 10.13189/ms.2022.100605
33. Jamal Salah, Hameed Ur Rehman, and, Iman Al- Buwaiqi, The Non-Trivial Zeros of the Riemann Zeta Function through Taylor Series Expansion and Incomplete Gamma Function, Mathematics and Statistics, vol. 10, no. 2, pp. 410-418, 2022, DOI: 10.13189/ms.2022.100216.
34. Jamal Salah, Some Remarks and Propositions on Riemann Hypothesis, Mathematics and Statistics, vol. 9, no. 2, pp. 159-165, 2021, DOI: 10.13189/ms.2021.090210.
35. Hameed Ur Rehman, Malina Darus, and Jamal Salah, Generalizing Certain Analytic Functions Correlative to the n-th Coefficient of Certain Class of Bi-Univalent Functions, Hindawi Journal of Mathematics Volume 2021, Article ID 6621315, 14 pages DOI: 10.1155/2021/6621315
36. Salah, J., (2021). Applications of differential Subordination and Superordination Involving Certain Fractional Operator. Current Topics on Mathematics and Computer Science, vol. 1, pp. 7–16, 2021 DOI: 10.9734/bpi/ctmcs/v1/9050D
37. Jamal Y. Mohammad Salah, The Consequence of the Analytic Continuity of Zeta Function Subject to an Additional Term and a Justification of the Location of the Non-Trivial Zeros, International Journal of Science and Research (IJSR), Vol. 9, Issue 3, March 2020, pp. 1566-1569, March 2020, URL: https://www.ijsr.net/get_abstract.php?paper_id=SR20328201050

38. Jamal Y. Mohammad Salah, An Alternative perspective to Riemann Hypothesis, PSYCHOLOGY AND EDUCATION, vol. 57, no. 9, pp. 1278-1281, 2020, URL: file:///C:/Users/drjamals/Downloads/454-Article%20Text-738-1-10-20210128%20(1).pdf
39. Jamal Y. Mohammad Salah, Two Conditional proofs of Riemann Hypothesis, International Journal of Sciences: Basic and Applied Research (IJSBAR), vol. 49, no. 1, pp. 74-83, 2020, URL: https://gssrr.org/index.php/JournalOfBasicAndApplied/article/view/10720
40. Jamal Salah, TWO NEW EQUIVALENT STATEMENTS TO RIEMANN HYPOTHESIS, Far East Journal of Mathematical Sciences (FJMS) Volume 118, Issue1, 2019, Pages 1- 8. Sept 2019.
DOI: 10.17654/MS118010001
41. Jamal Y. Salah, A NEW SUBCLASS OF UNIVALENT FUNCTIONS DEFINED BY THE MEANS OF JAMAL OPERATOR, Far East Journal of Mathematical Sciences (FJMS) Vol. 108, No. 2, 2018, pp. 389-399, 2018, Dec 2018, DOI: 10.17654/MS108020389.
42. Hameed Ur Rehman, Maslina Darus and **Jamal Salah**, Graphing Examples of Starlike and Convex Functions of order β, Appl. Math. Inf. Sci. vol. 12, No. 3, pp. 509-515, 2018, DOI :10.18576/amis/120305
43. Jamal Salah, Hameed Ur Rehman and Maslina Darus, A Note on Caputo's derivative interpretation in Economy, J. Appl. Math, pp. 1-7, 2018, DOI: 10.1155/2018/1260240.
44. Hameed Ur Rehman, Maslina Darus and Jamal Salah, Normalization of the generalized K− Mittag-Leffler function and ratio to its sequence of partial sums, BSG Proceedings, vol. 25, 2018, pp. 78-94.
45. Hameed Ur Rehman, Maslina Darus and Jamal Salah, COEFFICIENT PROPERTIES INVOLVING THE GENERALIZED K−MITTAG-LEFFLER FUNCTIONS, Trans. J. Math. Mecha. (TJMM) 9 (2017), No. 2, 155-164
46. Jamal Y. Salah. Closed-to-Convex Criterion Associated to the Modified Caputo's fractional Calculus Derivative Operator. Far East Journal of Mathematical Sciences (FJMS). Vol. 101, No. 1, pp. 55-59, 2017, DOI: 10.17654/MS101010055.
47. Jamal Y. Salah. Properties of the Modified Caputo's Derivative Operator for certain analytic functions. International Journal of Pure and Applied Mathematics. September 2016, Vol. 109, No. 3, pp. 665 – 671, 2014, DOI: 10.12732/ijpam. v109i3.14
48. Jamal Salah. A Note on the Modified Caputo's Derivative Operator. Far East Journal of Mathematical Sciences (FJMS). Vol. 100, No. 4, pp. 609-615, 2016,

DOI: 10.17654/MS100040609
49. S. Venkatesh, Jamal Salah, G. Sethuraman. "Some Results on E – Cordial Graphs", *International Journal of Mathematical Trends and Technology (IJMTT).* V7:121-125 March 2014.
DOI: 10.14445/22315373/IJMTT-V7P516
50. Jamal Y. Salah, A Note on Gamma Function, International Journal of Modern Sciences and Engineering Technology (IJMSET), vol. 2, no. 8, pp. 58-64, 2015,
51. Jamal Y. Salah, On Riemann Hypothesis and Robin's Inequality, International Journal of Scientific and Innovative Mathematical Research (IJSIMR), vol. 3, no. 4, pp. 9-14, April 2015,
URL: https://www.arcjournals.org/pdfs/ijsimr/v3-i4/4.pdf
52. Jamal Y. Salah, **A Note on Riemann Zeta Function,** IOSR Journal of Engineering (IOSRJEN), vol. 06, no. 02, pp. 07-16, February 2016,
URL: http://iosrjen.org/Papers/vol6_issue2%20(part-3)/B06230716.pdf

53. Jamal Y. Salah, A NOTE ON THE HURWITZ ZETA FUNCTION, Far East Journal of Mathematical Sciences (FJMS), vol. 101, no. 12, pp. 2677-2683, June 2017,
DOI: 10.17654/MS101122677
54. Jamal Salah and Venkatesh Srivastava. Inequalities on the Theory of Univalent Function. Journal of Mathematics and System Science, vol. 4, pp. 509-513, 2014.

DOI: 10.17265/2159-5291/2014.07.008
55. Jamal Salah. Neighborhood of a certain family of multivalent functions with negative coefficients. International Journal of Pure and Applied Mathematics. (IJPAM). Vol. 92, No 4, April 2014,
DOI: 10.12732/ijpam. v92i4.14
56. Jamal Salah. Fekete-Szego problems involving certain integral operator. International Journal of Mathematics Trends and Technology. IJMTT, vol. 7, no. 1, pp. 54-60, 2014.
DOI: 10.14445/22315373/IJMTT-V7P508

57. Jamal Salah. Subordination and superordination involving certain fractional operator. Asian Journal of Fuzzy and Applied Mathematics, vol. 1, pp. 98-107, 2013.
URL: https://www.ajouronline.com/index.php/AJFAM/article/view/724

58. Jamal Salah. Certain subclass of analytic functions associated with fractional calculus operator. Trans. J. Math. Mecha, vol. 3, no. 1, pp. 35-42, 2011.
URL: http://tjmm.edyropress.ro/journal/11030106.pdf

59. Jamal Salah. A note on Starlike functions of order α associated with a fractional calculus operator involving Caputo's fractional. J. Appl. comp Sc. Math, vol. 5, no. 1, pp. 97- 101, 2011,
URL: https://www.jacsm.ro/view/?pid=10_16

60. Jamal Salah and Maslina Darus. On convexity of general integral operators. J. Anale. Universitatii. De Vest. Timisoara. Seria Matematica – Informatica XLIX. 1 (2011) 117-124.

61. Jamal Salah and Maslina Darus. A note on generalized Mittag-Leffler and Application. Far East. Math Sc. (FJMS), vol. 48, no. 1, pp. 33-46, 2011.
URL: http://www.pphmj.com/abstract/5478.htm

62. Jamal Salah and Maslina Darus. A subclass of uniformly convex functions associated with fractional calculus operator involving Caputo's fractional differentiation. Acta. Univ. APL. Vol. 24, pp. 295-306, 2010.
63. D. G. Cantor, "Power series with integral coefficients," Bulletin of the American Mathematical Society, vol. 69 , no. 3, pp. $362-367,1963$.
64. R. Vein and P. Dale, "Determinants and Their Applications in Mathematical Physics," in Applied Mathematical Sciences, vol. 134, Springer, New York, 1999.
65. R. Wilson, "Determinantal criteria for meromorphic functions," Proceedings of the London Mathematical Society, vol. s3-4, no. 1, pp. 357-374, 1954.
66. C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," Journal of the London Mathematical Society, vol. s1-41, no. 1, pp. 111-122, 1966. [5] C.

Pommerenke, "On the Hankel determinants of univalent functions," Mathematika, vol. 14, no. 1, pp. 108-112, 1967.
67. D. Bansal, "Upper bound of second Hankel determinant for a new class of analytic functions," Applied Mathematics Letters, vol. 26, no. 1, pp. 103-107, 2013.
68. M. M. Elhosh, "On the second Hankel determinant of univalent functions," Bull. Malays. Math. Soc., vol. 9, no. 1, pp. 23-25, 1986.
69. J. W. Noonan and D. K. Thomas, "On the Hankel determinants of areally mean p-valent functions," Proceedings of the London Mathematical Society, vol. s3-25, no. 3, pp. 503-524, 1972.
70. J. W. Noonan, "Coefficient differences and Hankel determinants of areally mean p-valent functions," Proceedings of American Mathematical Society, vol. 46, pp. 29-37, 1974.
71. J. W. Noonan and D. K. Thomas, "On the second Hankel determinant of areally mean p-valent functions," Transactions of the American Mathematical Society, vol. 223, pp. 337-346, 1976.
72. M. M. Elhosh, "On the second Hankel determinant of close-toconvex functions," Bulletin of the Malaysian Mathematical Sciences Society, vol. 9, no. 2, pp. 67-68, 1986.
73. M. Arif, K. I. Noor, and M. Raza, "Hankel determinant problem of a subclass of analytic functions," Journal of Inequalities and Applications, vol. 2012, Article ID 22, 2012
74. T. Hayami and S. Owa, "Generalized Hankel determinant for certain classes," International Journal of Mathematics and Analysis, vol. 4, no. 49-52, pp. 2573-2585, 2010.
75. T. Hayami and S. Owa, "Applications of Hankel determinant for p-valently starlike and convex functions of order a," Far East Journal of Applied Mathematics, vol. 46, no. 1, pp. 1-23, 2010.
76. T. Hayami and S. Owa, "Hankel determinant for p-valently starlike and convex functions of order α, General Math, vol. 17 , no. 4 , pp. 29-44, 2009 .
77. A. Janteng, S. A. Halim, and M. Darus, "Hankel determinant for starlike and convex functions," International Journal of Mathematics and Analysis, vol. 1, no. 13-16, pp. 619-625, 2007.
78. A. K. Mishra and P. Gochhayat, "Second Hankel determinant for a class of analytic functions defined by fractional derivative," International Journal of Mathematics and Mathematical Sciences, vol. 2008, Article ID 153280, 10 pages, 2008.
79. N. Mohamed, D. Mohamad, and S. Cik Soh, "Second Hankel determinant for certain generalized classes of analytic functions," International Journal of Mathematics and Analysis, vol. 6, no. 17-20, pp. 807-812, 2012 .
80. G. Murugusundaramoorthy and N. Magesh, "Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant," Bulletin of Mathematical Analysis and Applications, vol. 1, no. 3, pp. 85-89, 2009.
81. K. I. Noor, "Higher order close-to-convex functions," Japanese Journal of Mathematics, vol. 37, no. 1, pp. 1-8, 1992.
82. K. I. Noor, "On the Hankel determinant problem for strongly close-to-convex functions," Journal of Natural Geometry, vol. 11 , no. 1, pp. 29-34, 1997.
83. K. I. Noor, "On certain analytic functions related with strongly close-to-convex functions," Applied Mathematics and Computation, vol. 197, no. 1, pp. 149-157, 2008. [23] K. I. Noor and S. A. Al-Bany, "On Bazilevic functions," International Journal of Mathematics and Mathematical Sciences, vol. 10, no. 1, 88 pages, 1987.
84. K. I. Noor, "On analytic functions related with functions of bounded boundary rotation," Commentarii Mathematici Universitatis Sancti Pauli, vol. 30, no. 2, pp. 113-118, 1981.

85. K. I. Noor, "On meromorphic functions of bounded boundary rotation," Caribb. J. Math., vol. 1, no. 3, pp. 95-103, 1982.
86. K. I. Noor, "Hankel determinant problem for the class of functions with bounded boundary rotation," Revue Roumaine de Mathématiques Pures et Appliquées, vol. 28, no. 8, pp. 731739,1983.
87. K. I. Noor and I. M. A. Al-Naggar, "On the Hankel determinant problem," Journal of Natural Geometry, vol. 14, no. 2, pp. 133-140, 1998.
88. R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, "The Fekete-Szegö coefficient functional for transforms of analytic functions," Bulletin of the Iranian Mathematical Society, vol. 35, no. 2, pp. 119-142, 2009.
89. R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Coefficient bounds for p_-valent functions," Applied Mathematics and Computation, vol. 187, no. 1, pp. 35-46, 2007. W. C. Ma and D. Minda, "A united treatment of some special classes of univalent functions," in Proceedings of the Conference on Complex Analysis, Tianjin, 1992.
90. R. K. Raina and J. Sokół, "On coefficient estimates for a certain class of starlike functions, Hacet," Journal of Mathematics and Statistics, vol. 44, no. 6, pp. 1427-1433, 2015.
91. G. Murugusundaramoorthy and T. Bulboacă, "Hankel determinants for new subclasses of analytic functions related to a shell shaped region," Mathematics, vol. 8, no. 6, article 1041, 2020.
92. S. Gandhi, "Radius Estimates for Three Leaf Function and Convex Combination of Starlike Functions," in Mathematical Analysis I: Approximation Theory. ICRAPAM 2018, vol. 306 of Springer Proceedings in Mathematics & Statistics, Springer, Singapore.
93. S. N. Malik, S. Mahmood, M. Raza, S. Farman, and S. Zainab, "Coefficient inequalities of functions associated with petal type domains," Mathematics, vol. 6, no. 12, p. 298, 2018.
94. K. Sharma, N. K. Jain, and V. Ravichandran, "Starlike functions associated with a cardioid," Afrika Matematika, vol. 27, pp. 923-939, 2016.
95. V. S. Masih and S. Kanas, "Subclasses of starlike and convex functions associated with the Limaçon domain," Symmetry, vol. 12, no. 6, p. 942,2020.
96. Y. Yuzaimi, A. H. Suzeini, and A. B. Akbarally, "Subclass of starlike functions associated with a limacon," AIP Conference Proceedings, vol. 1974, article 030023, 2018
97. P. L. Duren, Univalent Functions. Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, 1983.
98. U. Grenander and G. Szegö, Toeplitz Forms and Their Applications. California Monographs in Mathematical Sciences, University of California Press, Berkeley, 1958.
99. M. Abramowitz and I. A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications Inc., New York, 1965.
100. A. A. Attiya, Some applications of Mittag-Leffler function in the unit disk, Filomat 30(7) (2016), 2075-2081. https://doi .org/10 . 2298/FIL1607075A
101. D. Bansal and J. K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, Complex Var. Elliptic Equ. 61(3) (2016), 338-350. https://doi.org/10.1080/17476933.2015. 1079628
102. T. Bulboacă and G. Murugusundaramoorthy, Univalent functions with positive coefficients involving Pascal distribution series, Commun. Korean Math. Soc. 35(3) (2020),867 – 877. https: //doi . org/10.4134/CKMS . c190413
103. N. E. Cho, S. Y. Woo and S. Owa, Uniform convexity properties for hypergeometric functions, Fract. Calc. Appl. Anal. 5(3) (2002), 303-313.
104. P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften Series 259 , Springer Verlag, New York, 1983. [7] M. El-Deeb, T.

Bulboacă and J. Dziok, Pascal distribution series connected with certain subclasses of univalent functions, Kyungpook Math. J. $\mathbf{59}$ (2019), 301-314. https://doi.org/10.5666/ KMJ . 2019.59.2.301
105. B. A. Frasin, An application of an operator associated with generalized Mittag-Leffler function, Konuralp J. Math. 7(1) (2019), 199-202.
106. B. A. Frasin, T. Al-Hawary and F. Yousef, Some properties of a linear operator involving generalized Mittag-Leffler function, Stud. Univ. Babeş-Bolyai Math. 65(1) $(2020), 67-75$. https: //doi . org/10.24193/subbmath.2020.1.06
107. B. A. Frasin, T. Al-Hawary and F. Yousef, Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions, Afr. Mat. 30(1-2) (2019), 223-230. https: //doi.org/10.1007/s13370-018-0638-5
108. H. J. Haubold, A. M. Mathai and R. K. Saxena, Mittag-Leffler functions and their applications, J. Appl. Math. 2011(2011), Article ID 298628. https://doi.org/10.1155/2011/298628
109. V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series 301, Longman Scientific & Technical, Harlow, John Wiley & Sons, Inc., New York, 1994.
110. E. Merkes and B. T. Scott, Starlike hypergeometric functions, Proc. Amer. Math. Soc. 12 (1961), 885-888.
111. G. M. Mittag-Leffler, Sur la nouvelle fonction $\mathbf{E}(x)$, C. R. Acad. Sci. Paris $137(1903), 554-558$.
112. G. Murugusundaramoorthy, Subordination results for spiral-like functions associated with the Srivastava-Attiya operator, Integral Transforms Spec. Funct. 23(2) (2012), 97-103. https: //doi. org/10.1080/10652469.2011.562501
113. G. Murugusundaramoorthy, D. Răducanu and K. Vijaya, A class of spirallike functions defined by Ruscheweyh-type q-difference operator, Novi Sad J. Math. $\mathbf{49}(2)(2019), 59-71$. https://doi. org/10.30755/NSJOM.08284
114. G. Murugusundaramoorthy, K. Vijaya and S. Porwal, Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series, Hacet. J. Math. Stat. 45(4) (2016), 1101-1107. https: //doi.org/10.15672/HJMS . 20164513110
115. G. Murugusundaramoorthy, Subclasses of starlike and convex functions involving Poisson distribution series, Afr. Mat. 28(2017), 1357-1366. https://doi.org/10.1007/s13370-017-0520-x
116. S. Porwal and M. Kumar, A unified study on starlike and convex functions associated with Poisson distribution series, Afr. Mat. $27(5)$ (2016), 1021-1027. https://doi.org/10.1007/ s13370-016-0398-z
117. M. S. Robertson, On the theory of univalent functions, Ann. of Math. (2) 37(2) (1936), 374-408.
118. Salah, Jamal, Hameed Ur Rehman, and Iman Al-Buwaiqi. "The Non-Trivial Zeros of the Riemann Zeta Function through Taylor Series Expansion and Incomplete Gamma Function." *Mathematics and Statistics* 10.2 (2022): 410-418.
119. Rehman, Hameed Ur, Maslina Darus, and Jamal Salah. "Graphing Examples of Starlike and Convex Functions of order β." *Appl. Math. Inf. Sci* 12.3 (2018): 509-515.
120. Salah, Jamal Y., and O. M. A. N. Ibra. "Properties of the Modified Caputo's Derivative Operator for certain analytic functions." *International Journal of Pure and Applied Mathematics* 109.3 (2014): 665-671.
121. Jamal Salah and Maslina Darus, On convexity of the general integral operators, An. Univ. Vest Timis. Ser. Mat. -Inform. 49(1) (2011), 117-124.

122. Salah, Jamal. "TWO NEW EQUIVALENT STATEMENTS TO RIEMANN HYPOTHESIS." (2019).
123. H. Silverman, Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl. 172 (1993), 574-581. https://doi . org/10.1006/jmaa.1993. 1044
124. H. M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, Hypergeometric functions in the parabolic starlike and uniformly convex domains, Integral Transforms Spec. Funct. 18 (2007),$511-520$. https://doi.org/10.1080/10652460701391324
125. L. Spaček, Contribution à la théorie des fonctions univalentes, Časopis Pro Pěstování Matematiky $\mathbf{62}(1932), 12-19$.
126. Salah, Jamal Y. "A new subclass of univalent functions defined by the means of Jamal operator." *Far East Journal of Mathematical Sciences (FJMS) Vol* 108 (2018): 389-399.
127. Jamal Y. Salah On Riemann Hypothesis and Robin's Inequality. International Journal of Scientific and Innovative Mathematical Research (IJSIMR). Volume (3) 4 (2015) 9-14.
128. Salah, Jamal. "Neighborhood of a certain family of multivalent functions with negative coefficients." *Int. J. Pure Appl. Math* 92.4 (2014): 591-597.
129. Salah, Jamal, and Maslina Darus. "A note on Starlike functions of order α associated with a fractional calculus operator involving Caputo's fractional." *J. Appl. comp Sc. Math* 5.1 (2011): 97-101.
130. J. Salah, Certain subclass of analytic functions associated with fractional calculus operator, Trans. J. Math. Mech., 3 (2011), 35–42. Available from: http://tjmm.edyropress.ro/journal/11030106.pdf.
131. Salah, Jamal. "Some Remarks and Propositions on Riemann Hypothesis." *Mathematics and Statistics* 9.2 (2021): 159-165.
132. Salah, Jamal Y. Mohammad. "The consequence of the analytic continuity of Zeta function subject to an additional term and a justification of the location of the non-trivial zeros." *International Journal of Science and Research (IJSR)* 9.3 (2020): 1566-1569.
133.
134. A. Swaminathan, Certain sufficient conditions on Gaussian hypergeometric functions, Journal of Inequalities in Pure and Applied Mathematics $5(4)$ (2004), Article ID 83, 10 pages.
135. A. Wiman, Über die nullstellen der funktionen $\mathbf{E}_{\alpha}(x)$, Acta Math. 29 (1905), 217-134.
136. Salah, Jamal Y. Mohammad. "An Alternative perspective to Riemann Hypothesis." *PSYCHOLOGY AND EDUCATION* 57.9 (2020): 1278-1281.
137. Salah, Jamal Y. "A note on the Hurwitz zeta function." *Far East Journal of Mathematical Sciences (FJMS)* 101.12 (2017): 2677-2683.
138. Jamal Y. Salah. Closed-to-Convex Criterion Associated to the Modified Caputo's fractional Calculus Derivative Operator. Far East Journal of Mathematical Sciences (FJMS). Vol. 101, No. 1, pp. 55-59, 2017, DOI: 10.17654/MS101010055.
139. Jamal Y. Salah, A Note on Riemann Zeta Function, IOSR Journal of Engineering (IOSRJEN), vol. 06, no. 02, pp. 07-16, February 2016, URL: http://iosrjen.org/Papers/vol6_issue2%20(part-3)/B06230716.pdf

140. Jamal Y. Salah, A Note on Gamma Function, International Journal of Modern Sciences and Engineering Technology (IJMSET), vol. 2, no. 8, pp. 58-64, 2015.

141. Salah, Jamal, and S. Venkatesh. "Inequalities on the Theory of Univalent Functions." *Journal of Mathematics and System Science* 4.7 (2014).

142. T. R. Prabhakar, A single integral equation with a generalized Mittag – Leffler function in the kernel, Yokohama Math. J. 19 (1971), 7 – 15.

143. Omar Jawabreh, Ahmad Abdel Qader, Jamal Salah, Khaled Al Mashrafi, Emad Al Dein AL Fahmawee, Basel J. A. Ali, Fractional Calculus Analysis of Tourism Mathematical Model, Progr. Fract. Differ. Appl. Vol. 9, No. S1, pp. 1 -11 202) DOI: 10.18576/pfda/09S101 URL:

144. Jamal Salah. Subordination and superordination involving certain fractional operator. Asian Journal of Fuzzy and Applied Mathematics, vol. 1, pp. 98-107, 2013. URL: https://www.ajouronline.com/index.php/AJFAM/article/view/724

145. Salah, Jamal Y. Mohammad. "Two Conditional proofs of Riemann Hypothesis." *International Journal of Sciences: Basic and Applied Research (IJSBAR)* 49.1 (2020): 74-83.

146. Salah, Jamal. "Fekete-Szego problems involving certain integral operator." *International Journal of Mathematics Trends and Technology-IJMTT* 7 (2014).